Formalization of the First Theorem of Welfare Economics

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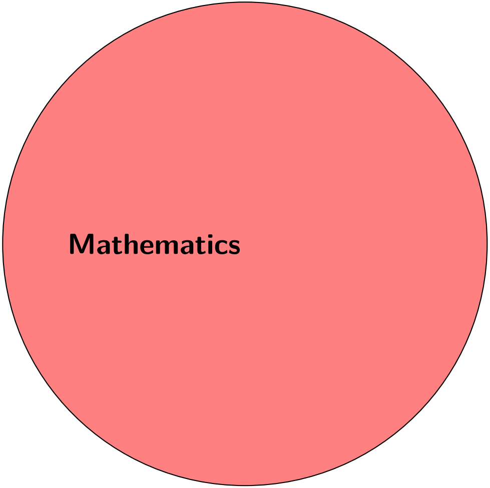
Section 1

Economics

Where are we?

Where are we?



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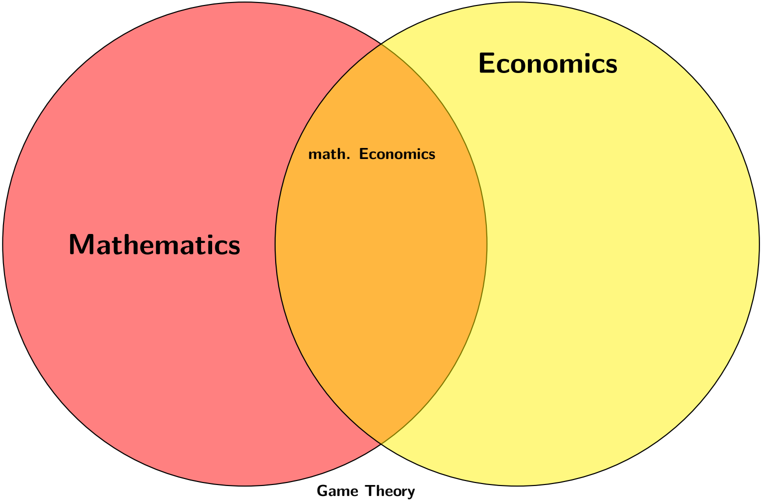

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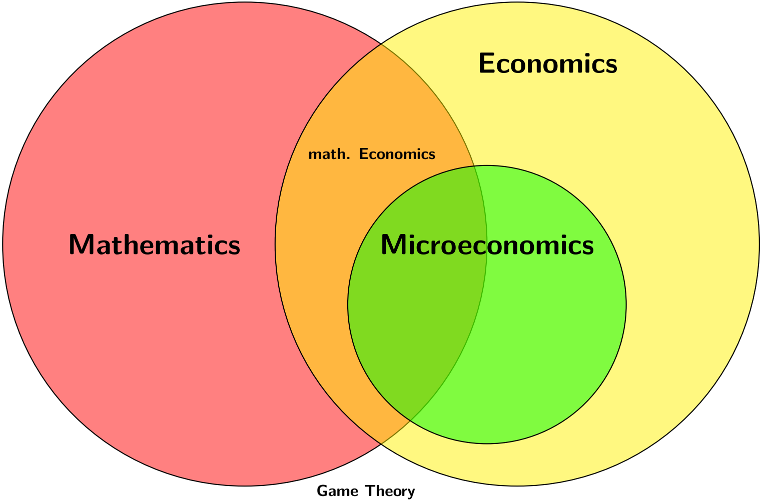
Economics
math. Economics
Mathematics
Microeconomics
Game Theory

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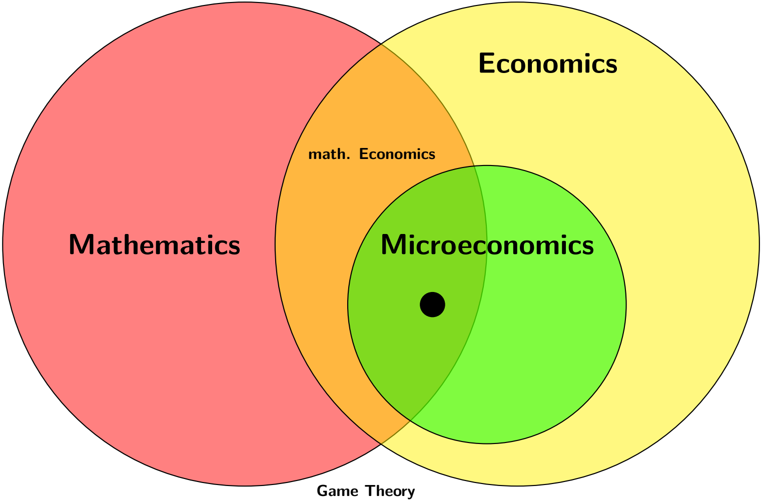
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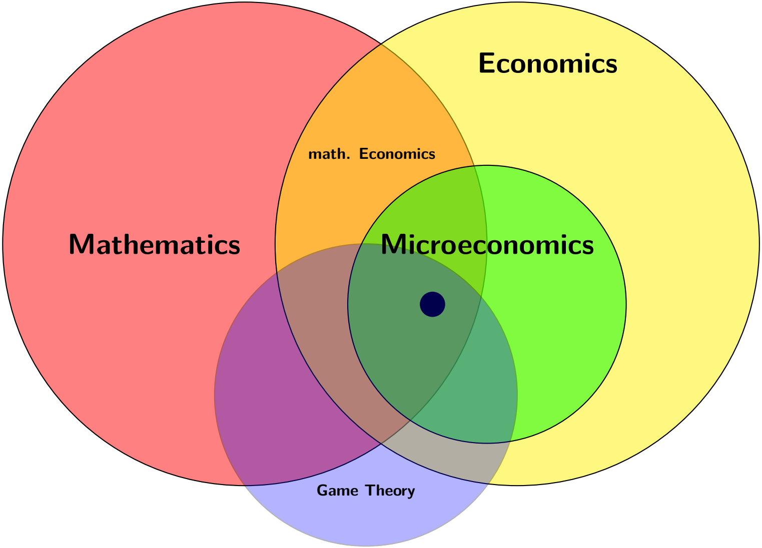
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Economy vs Market

Economy vs Market

Economy

- more general notion of a game.
- "Set of Markets"
- Market Economy, Planned Economy, Command Economy, etc.

Economy vs Market

Economy

- more general notion of a game.
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Market

- "A place for transaction"
- "Rules of transaction"
- Competitive markets, duopoly, monopoly, etc.

The Agent

The Agent

Consumption sets

- "Set of Goods and Services"
- represented as vectors: $(1, 0, 12, \pi, 0, \ldots, 5)$
- *n* goods represented by an *n*-dimensional euclidean space

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Comparison

- Preference (relation)
- Utility function

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- represented as vectors: $(1, 0, 12, \pi, 0, \ldots, 5)$
- n goods represented by an n-dimensional euclidean space

Comparison

- Preference (relation)
- Utility function

Definition (Utility function)

Given a preference relation $\succeq$, a utility function u, is defined:

$$u \; : \; X^n \mapsto \mathbb{R}$$
$$\forall x\ y\ .\ x \succeq y \iff u(x) \geq u(y).$$

(1 apple, 1 orange) $\succ$ (0, 0);

(1 apple, 1 orange) $\succ$ (0, 0); (1, 0) $\succ$ (0, 1);

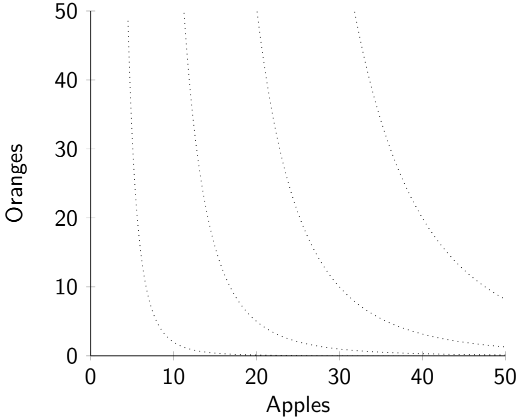
u(1, 1) $>$ u(0, 0); u(1, 0) $>$ u(0, 1);

(1 apple, 1 orange) $\succ$ (0, 0); (1, 0) $\succ$ (0, 1); (10, 5) $\approx$ (9, 10)

u(1, 1) $>$ u(0, 0); u(1, 0) $>$ u(0, 1); u(10, 5) $=$ u(9, 10)

$(1 \text{ apple}, 1 \text{ orange}) \succ (0, 0); (1, 0) \succ (0, 1); (10, 5) \approx (9, 10)$

$u(1, 1) > u(0, 0);\ u(1, 0) > u(0, 1); u(10, 5) = u(9, 10)$



Pareto Efficiency & Walrasian Equilibrium

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Definition (Walrasian Equilibirum)

An allocation and a price is said to be in a Walrasian Equilibrium if every consumer chooses bundles that maximizes their utility while being subject to the budget constraint.

Pareto Efficiency & Walrasian Equilibrium

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An allocation and a price is said to be in a Walrasian Equilibrium if every consumer chooses bundles that maximizes their utility while being subject to the budget constraint.

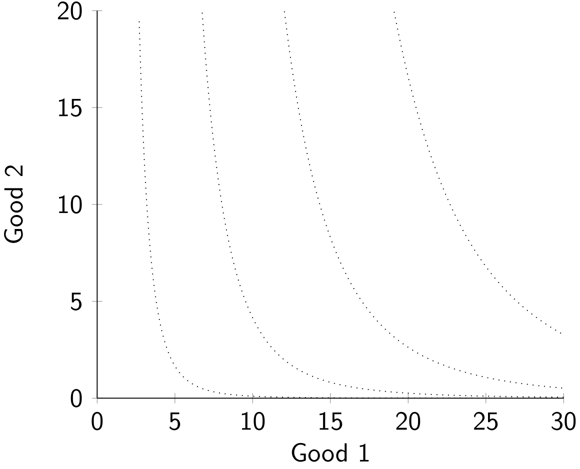
Definition (Pareto Efficiency)

Pareto efficiency is said to occur when it is impossible to make one agent better off without making another worse off.

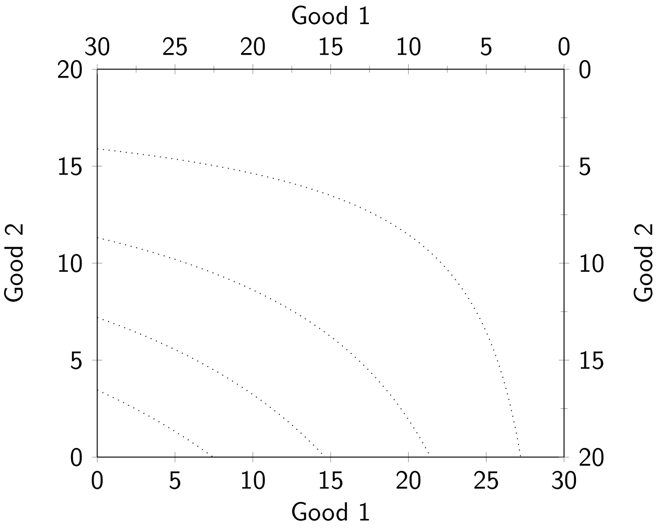
Edgeworth Box

- Pure exchange Market Model
- 2 consumers
- 2 goods
- Visual Interpretation

Edgeworth Box Economy



Edgeworth Box Economy



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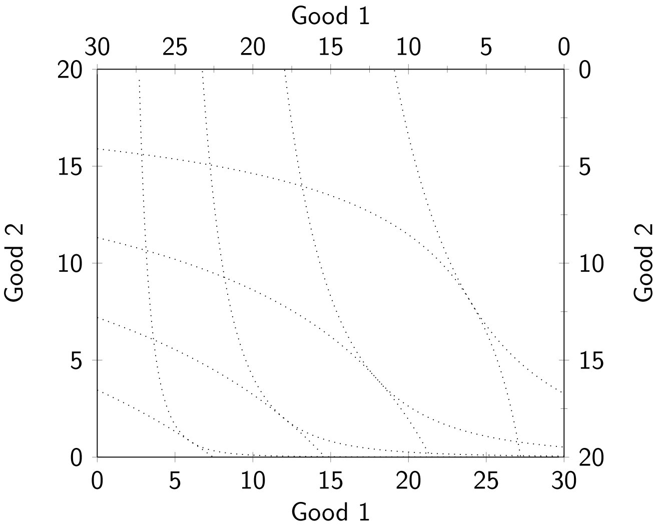

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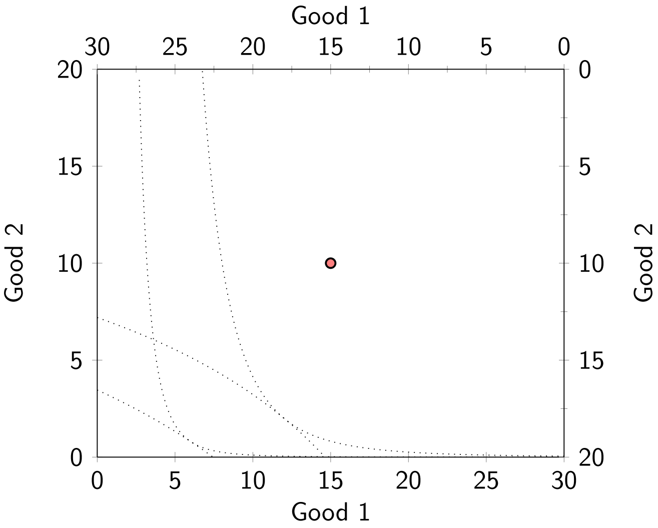

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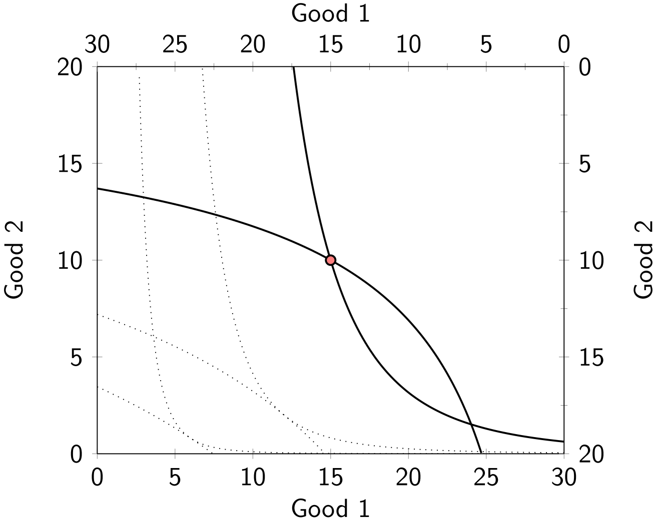

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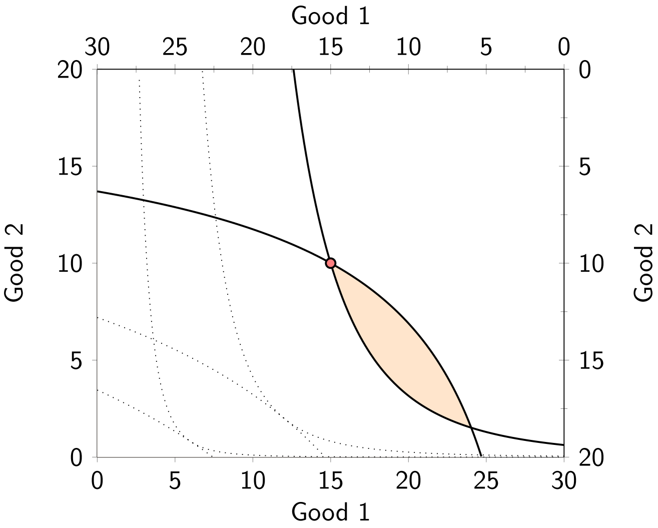

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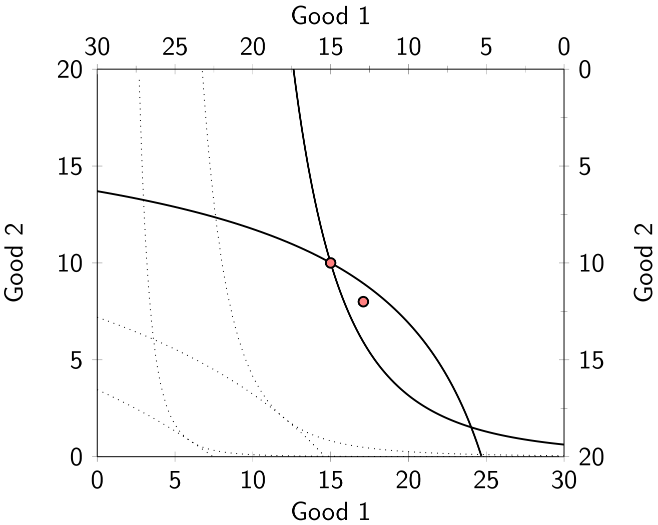

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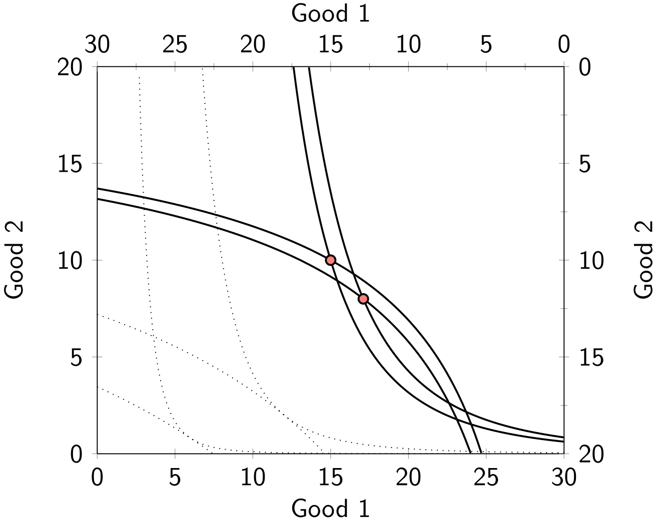

Section 2

Formalization

Utility and Preference

Preference relation

```
locale preference =
  fixes carrier :: "'a set"
  fixes relation :: "'a relation"
  assumes not_outside: "(x,y) ∈ relation ⟹ x ∈ carrier"
    and "(x,y) ∈ relation ⟹ y ∈ carrier"
  assumes trans_refl: "preorder_on  carrier relation"
```

Rational preference relation

```
locale rational_preference = preference +
  assumes "total_on carrier relation"
```

Utility and Preference cont.

Utility function

```
locale ordinal_utility =
  fixes carrier :: "'a set"
  fixes relation :: "'a relation"
  fixes u :: "'a ⇒ real"
  assumes "x ∈ carrier ⟹ y ∈ carrier ⟹
      x ≽[relation] y ⟷ u x ≥ u y"
  assumes "x ≽[relation] y ⟹ x ∈ carrier"
    and "x ≽[relation] y ⟹ y ∈ carrier"
```

Local non-satiation and Pareto ordering

Local non-satiation

```
definition local_nonsatiation where
  "local_nonsatiation B P ⟷ (∀x∈B. ∀e>0. ∃y∈B.
    norm (y - x) ≤ e ∧ y ≻[P] x)"
```

Local non-satiation and Pareto ordering

Local non-satiation

definition local_nonsatiation **where**
"local_nonsatiation B P $\longleftrightarrow$ ($\forall$x$\in$B. $\forall$e>0. $\exists$y$\in$B.
norm (y - x) $\leq$ e $\wedge$ y $\succ$[P] x)"

Pareto ordering

definition pareto_dominating **where**
"X $\succ$Pareto Y $\longleftrightarrow$
($\forall$i $\in$ agents. U[i] (X i) $\geq$ U[i] (Y i)) $\wedge$
($\exists$i $\in$ agents. U[i] (X i) > U[i] (Y i))"

Section 3

Models

Exchange economy

```
locale exchange_economy =
  fixes consumption_set :: "('a::ordered_euclidean_space) set"
  fixes agents :: "'i set"
  fixes ℰ :: "'i ⇒ 'a"
  fixes Pref :: "'i ⇒ 'a relation"
  fixes U :: "'i ⇒ 'a ⇒ real"
  fixes Price :: "'a"
  assumes "Price > 0"
  assumes "i ∈ agents ⟹
   eucl_ordinal_utility consumption_set (Pref i) (U i)"
  assumes "finite agents" and "agents ≠ {}"
```

Exchange economy

```
...
fixes firms :: "'f set"
fixes Θ :: "'i ⇒ 'f ⇒ nat"  ("Θ[_,_]")
assumes "i ∈ agents ⟹
  eucl_ordinal_utility consumption_set Pr[i] U[i]"
and "pre_arrow_debreu_consumption_set consumption_set"
assumes "j ∈ firms ⟹ (∑ i∈agents. Θ[i,j]) = 1"
assumes "Price > 0"
assumes "finite agents" and "agents ≠ {}"
```

Competitive Equilibria

```
definition competitive_equilibrium
  where
    "competitive_equilibrium P X Y ⟷ feasible X Y ∧
    (∀j ∈ firms. (Y j) ∈ profit_maximisation (production_sets j)) ∧
    (∀i ∈ agents. (X i) ∈ arg_max_set U[i]
      (budget_constraint (poe_wealth i Y)))"
```

```
definition budget_constraint
  where
    "budget_constraint W =
        {x ∈ consumption_set. Price · x ≤ W}"
```

Section 4

The First Welfare Theorem

Some History

Some History

> *It is not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard to their own interest.*
>
> — Adam Smith, Wealth of Nations (1776)

> *By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it.*
>
> — Adam Smith, Wealth of Nations (1776)

First Welfare Theorem

Theorem (First Theorem of Welfare Economics)

Assuming locally non-satiated preferences for each agent, any allocation in combination with a price vector that forms a Walrasian Equilibrium is Pareto Efficient.

First Welfare Theorem

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Assuming locally non-satiated preferences for each agent, any allocation in combination with a price vector that forms a Walrasian Equilibrium is Pareto Efficient.

theorem first_welfare_theorem_exchange:
  assumes "$\bigwedge$i. i $\in$ agents $\Longrightarrow$
    local_nonsatiation consumption_set Pr[i]"
  assumes "comp_equilib_endow Price $\mathcal{X}$ $\mathcal{E}$"
  shows "pareto_optimal_endow $\mathcal{X}$ $\mathcal{E}$"

Future Work and Misc.

- Second Welfare Theorem
- more (economic) models
- formalizing more Game Theory: Algorithmic Game theory, Mechanisms, . . .

https://www.isa-afp.org/entries/First_Welfare_Theorem.html

Questions?

Ordinal Utility

Ordinal Utility

lemma ordinality_of_utility_function :
  fixes f :: "real ⇒ real"
  assumes monot: "monotone (op>) (op>) f"
  shows "(f ∘ u) x > (f ∘ u) y ⟷ u x > u y"

Finite Carrier

theorem *fnt_carrier_exists_util_fun:*
  assumes *"finite carrier"*
  assumes *"rational_preference carrier relation"*
  shows *"∃ u. ordinal_utility carrier relation u"*

Walras' Law

Walras' Law

```
lemma walras_law:
  assumes "⋀i. i∈agents ⟹ local_nonsatiation
consumption_set Pr[i]"
  assumes "competitive_equilibrium Price X Y"
  shows "Price • ((∑ i∈agents. (X i)) -
(∑ i∈agents. ℰ[i]) - (∑ j∈firms. Y j)) = 0"
```

Walras' Law